

# A counterexample to a geometric Hales-Jewett type conjecture

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## Abstract

Pór and Wood conjectured that for all  $k, l \geq 2$  there exists  $n \geq 2$  with the following property: whenever  $n$  points, no  $l + 1$  of which are collinear, are chosen in the plane and each of them is assigned one of  $k$  colours, then there must be a line (that is, a maximal set of collinear points) all of whose points have the same colour. The conjecture is easily seen to be true for  $l = 2$  (by the pigeonhole principle) and in the case  $k = 2$  it is an immediate corollary of the Motzkin-Rabin theorem. In this note we show that the conjecture is false for  $k, l \geq 3$ .

## 1 Introduction

Given a finite set  $S$  in the plane, we will use the term *line* to denote any maximal set of collinear points of  $S$ . Pór and Wood posed the following conjecture.

**Conjecture 1** (Pór and Wood [4]). *For all integers  $k \geq 1$  and  $l \geq 2$ , there is an integer  $n$  such that for every finite set  $S$  of size  $|S| \geq n$  in the plane  $\mathbb{R}^2$ , if each point of  $S$  is assigned one of  $k$  colours, then*

- *$S$  contains  $l + 1$  collinear points, or*
- *$S$  contains a monochromatic line.*

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The motivation for this conjecture comes from the Hales-Jewett theorem. By a *combinatorial line* in the grid  $[l]^n \subset \mathbb{R}^n$  (where  $[l]$  stands for the set  $\{1, 2, \dots, l\}$ ) we mean a set of the form

$$\{(x_1, \dots, x_n) \in [l]^n : x_i = x_j \text{ for all } i, j \in I\}$$

for fixed  $I \subset [n], I \neq \emptyset$  and fixed  $x_i, i \in [n] \setminus I$ . Now the Hales-Jewett theorem can be stated as follows.

**Theorem 2** (Hales and Jewett [1]). *For all integers  $k, l \geq 1$ , there is an integer  $n$  such that whenever each of the points in  $[l]^n \subset \mathbb{R}^n$  is given one of  $k$  colours, there is a monochromatic combinatorial line.*

Conjecture 1 is a natural geometric version of this theorem, where the lines are not necessarily parallel to a fixed set of axes, and the ambient set can be any set without many collinear points.

For  $l = 2$  the result is trivial: we may take  $n = k + 1$  and by the pigeonhole principle there is a line containing two points of the same colour. The case  $k = 2$  is a special case of the Motzkin-Rabin theorem that was proved in [3]. In this paper we demonstrate by a counterexample that the conjecture is false in the next smallest case  $k = l = 3$ , and hence it is false whenever  $k, l \geq 3$ .

**Theorem 3.** *For any  $n \geq 2$ , there is a set  $S \subset \mathbb{R}^2$  of size  $n$  satisfying:*

- *no four points of  $S$  are collinear, and*
- *the points of  $S$  can be coloured using three colours in such a way that no line is monochromatic.*

## 2 Proof of Theorem 3

We start by noting that it is sufficient to find a set with the required properties in the projective plane  $\mathbb{RP}^2$ . Indeed, given a finite set  $S \subset \mathbb{RP}^2$ , one can choose a line  $l \subset \mathbb{RP}^2$  that does not meet  $S$  and apply a projective transformation that sends  $l$  to the line at infinity. The image of  $S$  under this transformation is contained in the affine plane  $\mathbb{R}^2$  while the collinearity relations of the original set  $S$  are preserved.

Our counterexample is a finite subset of the irreducible cubic curve  $y^2 = x^3 - x^2$ . More specifically, we will use a subset of the set of its non-singular points  $\Gamma = \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 - x^2, x \neq 0\} \cup \{\mathcal{O}\} \subset \mathbb{RP}^2$  where  $\mathcal{O}$  is a point at infinity that is contained in all lines parallel to the  $y$ -axis and in

the line at infinity. By the Bézout theorem,  $\Gamma$  does not contain a set of four collinear points. Moreover, it is a well known fact in algebraic geometry that  $\Gamma$  forms an abelian group with the property that distinct points  $P, Q, R \in \Gamma$  are collinear if and only if  $P + Q + R = 0$ , and that  $\Gamma$  is isomorphic to the circle group  $\mathbb{R}/\mathbb{Z}$  (see [2], p. 19–20).

In fact, any choice of an elliptic curve whose group is isomorphic to  $\mathbb{R}/\mathbb{Z}$  would do. However, we choose this particular cubic curve (which is not an elliptic curve as it contains a singular point  $(0, 0)$ ) because it admits a simple explicit group isomorphism  $\phi : \mathbb{R}/\mathbb{Z} \rightarrow \Gamma$ , given by

$$\phi(x) = \begin{cases} (\cot(\pi x)^2 + 1, \cot(\pi x)(\cot(\pi x)^2 + 1)) & \text{if } x \neq 0, \\ \mathcal{O} & \text{if } x = 0. \end{cases}$$

This enables us to give a self-contained proof of the theorem without referring to any results from algebraic geometry. However, the reader familiar with elliptic curves can skip the proof of the following proposition.

**Proposition 4.** *Let  $x, y$  and  $z$  be distinct elements of  $\mathbb{R}/\mathbb{Z}$ . Then the points  $\phi(x), \phi(y)$  and  $\phi(z)$  are collinear if and only if  $x + y + z = 0$ . Moreover,  $\phi : \mathbb{R}/\mathbb{Z} \rightarrow \Gamma$  is a well defined bijection.*

*Proof.* The fact that  $\phi$  is a well defined bijection follows from the basic properties of the cotangent function. To prove the equivalence of the geometric and algebraic relations, we will use the identity

$$\cot(x + y) = \frac{\cot(x) \cot(y) - 1}{\cot(x) + \cot(y)}, \quad (1)$$

which holds whenever  $x + y, x, y$  are not multiples of  $\pi$ . Given a real number  $r \notin \mathbb{Z}$ , define  $c_r = \cot(\pi r)$ .

If one of  $x, y, z \in \mathbb{R}/\mathbb{Z}$  is 0 (say,  $x = 0$ ) then  $\phi(z)$  is collinear with  $\phi(x) = \mathcal{O}$  and  $\phi(y)$  if and only if  $\phi(z)$  is the reflection of  $\phi(y)$  in the  $x$ -axis, that is,  $z = -y$ . Similarly, if two of the numbers (say,  $x$  and  $y$ ) sum to 0, then the three points are collinear if and only if  $\phi(z) = \mathcal{O}$ , that is,  $z = 0$ . Now we can assume that  $x, y, z$  are all non-zero and that no two of them sum to 0. Then the points  $\phi(x), \phi(y)$  and  $\phi(z)$  are collinear if and only if

$$\frac{c_z(c_z^2 + 1) - c_x(c_x^2 + 1)}{(c_z^2 + 1) - (c_x^2 + 1)} = \frac{c_z(c_z^2 + 1) - c_y(c_y^2 + 1)}{(c_z^2 + 1) - (c_y^2 + 1)},$$

which after rearrangement becomes

$$c_z = -\frac{c_x c_y - 1}{c_x + c_y}.$$

Notice that  $z = -x - y$  is a solution by (1), and it is unique in  $\mathbb{R}/\mathbb{Z}$  as  $\cot$  is injective on  $(0, \pi)$ .  $\square$

Now we are ready to finish the proof of the theorem.

*Proof of Theorem 3.* As noted before, it is enough to construct a set  $S' \subset \mathbb{RP}^2$  with the two required properties, and take a projective transformation that maps  $S'$  into  $\mathbb{R}^2$ .

For the set  $S'$  (see Fig. 1 and 2) we will take  $S' = \{\phi(i/n) : i = 0, \dots, n-1\}$ . Notice that by Proposition 4 there are no four collinear points in  $S'$ . Indeed, if  $\phi(x), \phi(y), \phi(z)$  and  $\phi(w)$  were distinct and collinear, then  $z = -x - y = w$  in  $\mathbb{R}/\mathbb{Z}$ , giving a contradiction. Colour  $\phi(i/n)$

$$\begin{aligned} &\text{red if } 0 \leq i < \frac{n}{3}, \\ &\text{green if } \frac{n}{3} \leq i < \frac{2n}{3}, \\ &\text{blue if } \frac{2n}{3} \leq i < n. \end{aligned}$$

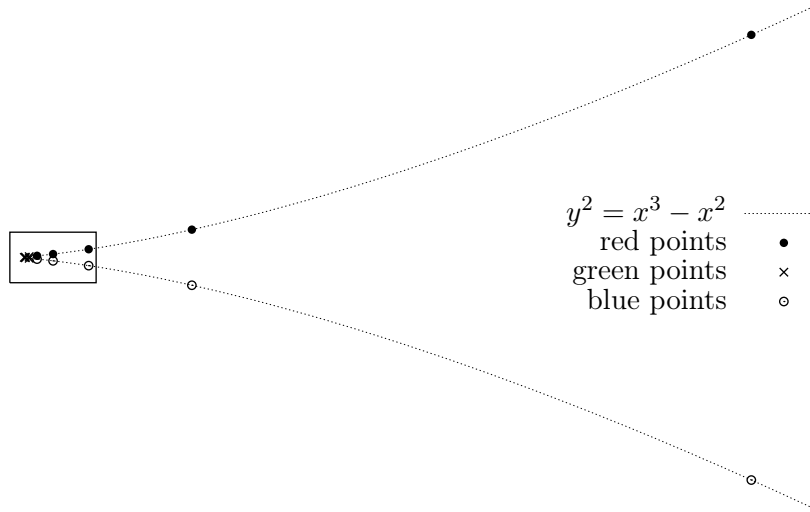


Figure 1: The set  $S'$  with  $n = 16$ . The sixteenth point is at infinity, and has red colour. The framed section is shown in smaller scale in Fig. 2.

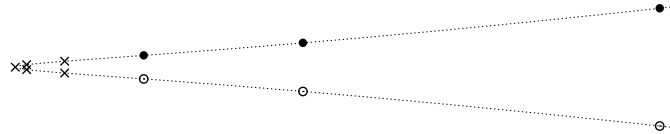


Figure 2: Part of the set  $S'$  with  $n = 16$  in smaller scale.

Suppose for contradiction that there is a monochromatic line  $l$ . It must pass through two distinct points  $\phi(i/n)$  and  $\phi(j/n)$ ,  $0 \leq i, j < n$ . There is an integer  $0 \leq k < n$  such that  $k \equiv -i - j \pmod{n}$ , possibly  $k = i$  or  $k = j$ . Then  $i/n + j/n + k/n = 0$  in  $\mathbb{R}/\mathbb{Z}$ , and so by Proposition 4 either  $\phi(i/n), \phi(j/n)$  and  $\phi(k/n)$  are distinct colinear points, or  $\phi(k/n)$  coincides with one of the other two points. In either case  $l$  passes through all of these points, and hence they have the same colour.

Now write  $i/n = x + \alpha, j/n = x + \beta$  and  $k/n = x + \gamma$ , where  $x \in \{0, \frac{1}{3}, \frac{2}{3}\}$  and  $\alpha, \beta, \gamma \in [0, \frac{1}{3})$ . Considered as real numbers,  $3x$  and  $i/n + j/n + k/n = 3x + \alpha + \beta + \gamma$  are integers, so  $\alpha + \beta + \gamma$  is also an integer. But  $0 \leq \alpha, \beta, \gamma < \frac{1}{3}$ , so this is only possible if  $\alpha = \beta = \gamma = 0$ . In particular,  $i/n = j/n$ , contradicting the assumption that  $\phi(i/n) \neq \phi(j/n)$ .

This finishes the proof.  $\square$

## References

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